Operations Research Letters 44 (2016) 775–778

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Information, egalitarianism and the value

R.R. Routledge

The University of Liverpool, United Kingdom

ARTICLE INFO

Article history:
Received 29 March 2016
Received in revised form
21 September 2016
Accepted 21 September 2016
Available online 3 October 2016

Keywords:
Missing information
Cooperative games
Egalitarianism
Nullifying players

ABSTRACT

This work proposes a value for a cooperative game with missing information. The value distributes the worth of the grand coalition on the basis of the number of known coalitions. When all the information is contained in the characteristic function the value coincides with the equal division solution. An axiomatic characterization of the value is presented which uses the nullifying player axiom introduced in van den Brink (2007).

1. Introduction

Despite it being over sixty years since it was first published, the most important single-valued solution for a cooperative game remains the Shapley value proposed in [11]. One important reason to explain this is that the Shapley value uses all the information contained in the characteristic function to determine each player's payoff. Also, the Shapley value admits a simple and convincing axiomatic characterization. Shapley himself expressed surprise at the characterization noting that "It is remarkable that no further conditions are required to determine the value uniquely". Another reason for the prominence of the Shapley value is the deep connections, explained in [1], between the Shapley value and the competitive equilibrium in smooth economies with an atomless continuum of individuals. However, because all the information about the worths of the coalitions is required to calculate the Shapley value, this is also a drawback of the solution. If the characteristic function represents a one-off interaction between the players it may be that there is no way of knowing the worths of particular coalitions. An alternative, which is easier to model using standard tools, is to consider the case where the characteristic function is uncertain. Many papers have suggested solutions to uncertain cooperative games, including [3,4,9,12]. Suppose, then, some information is missing from the characteristic function. How should an impartial observer distribute the worth of the grand coalition amongst the players? It is this question which this work tries to answer.

A cooperative game with missing information is a mapping from the power set of the players into the union of the real line and the empty set. If, for some coalition, their worth is equal to the empty set, then the worth of the coalition is treated as being missing from the game. Therefore, the model permits arbitrary information to be missing from the characteristic function. The only assumptions imposed upon the characteristic function are that the worth of the grand coalition is not equal to the empty set, and the worth of the empty set is zero. Ideally, one would like a value for a game to possess three key properties:

1. Be applicable to the whole domain under consideration.
2. Be easy to calculate—at least in games with small numbers of players.
3. It should have a clear axiomatic characterization which reveals the underlying behaviour of the value.

The solution advanced here satisfies these three properties. The value distributes the worth of the grand coalition between the players in proportion to the number of known coalitions of which each player is a member. It is such that when all the information is contained in the characteristic function it coincides with the equal division solution. Therefore, the value can be thought of as an extension of the equal division solution to a wider domain than is normally studied in the literature. An axiomatic characterization of the value is presented using four axioms: efficiency, additivity, nullifying player and known coalitions. Efficiency requires that the value always distributes the worth of the grand coalition amongst the players. The axiom of additivity states, in the usual way, that if the value is applied to the sum of games, then the payoff should be equal to the sum of the payoffs when the value is applied to the separate games. The axiom of nullifying player was used

E-mail address: R.R.Routledge@liverpool.ac.uk.

http://dx.doi.org/10.1016/j.orl.2016.09.010
0167-6377/

in [13] to highlight the differences between the Shapley value and egalitarian solutions. A player is a nullifying player if all the known coalitions which contain the player have a worth of zero. The axiom states that nullifying players receive zero payoff. Finally, the axiom of known information applies to non-nullifying players and requires that the payoffs be related to the number of known coalitions in the game. The reasoning behind the known coalitions axiom is that, if there is positive worth distributed amongst non-nullifying players, then the more known coalitions a player belongs to, the higher should be their payoff. When all the information is contained in the characteristic function, the axiom of known coalitions implies symmetric payoffs to non-nullifying players.

1.1. Related literature

The interest in axiomatic value theory began with [11] demonstrating that the axioms of efficiency, symmetry, additivity and carrier player characterize a unique solution on the space of cooperative games. Since that paper there has been sustained interest in axiomatic value theory. This brief survey only covers the most important, and recent contributions, to the literature. A succinct and detailed textbook treatment of value theory is contained in [7], especially chapters 18 and 20. In [10] the main alternative to the Shapley value was proposed, which is the nucleolus. The nucleolus lexicographically minimizes the ordered vector of excesses, and unlike the Shapley value, always selects from the core of a game whenever it is non-empty. Even if the core is empty, the nucleolus is often interpreted as the core-centre of a game. In a series of papers, [14,15] it was demonstrated that the mathematically convenient axiom of additivity could be dropped in the characterization of the Shapley value and replaced with the economically intuitive axiom of strong monotonicity (or marginalism).

In recent years, there has been particular interest in values which are more egalitarian than the Shapley value, or can be applied to wider domains. In [8] a solidarity value for a cooperative game was suggested. In this value, each player receives the average of the marginal contributions of the players preceding them in each permutation. By replacing the usual null player axiom with an average null player axiom, and finding a basis for a game which differed from the usual unanimity game basis, they were able to characterize the solidarity value. In [13] it was demonstrated that if the null player axiom is replaced with a nullifying player axiom in the usual Shapley characterization, then the axioms characterize the equal division solution (in which the worth of the grand coalition is divided equally between the players). Several other egalitarian solutions were also analysed, such as the centre of the imputation set, and an application to auction theory was presented. The work in [6] studied procedural values in which each player could receive particular shares of the marginal contributions of players preceding them in each permutation of the players. These procedural values include the Shapley value, the solidarity value and the equal division solution as special cases. In [5] the same model as in this paper was analysed, in which arbitrary information may be missing from a cooperative game, and considered three different ways in which the Shapley value may be extended to the missing information domain. In [2] a production economy was studied, and the axioms of efficiency, symmetry and monotonicity were shown to characterize overall proportional taxation rates.

2. The model

Let $N = \{1, \ldots, n\}$, $n \geq 2$, be the finite set of players in the game. A **cooperative game with missing information** is (N, v) with v being the characteristic function $v : 2^N \rightarrow \Re \cup \{\varnothing\}$. A **coalition** is an $S \subseteq N$. The **grand coalition** is N. For any coalition $S \subseteq N$, if $v(S) \neq \varnothing$, then $v(S)$ is the money **worth** which S can obtain independently of the other players. If $v(S) = \varnothing$, then the worth of coalition S is missing from the game.

Assumption 2.1. For every $v : 2^N \rightarrow \Re \cup \{\varnothing\}$, $v(N) \neq \varnothing$ and $v(\varnothing) = 0$.

Let G denote the space of games satisfying Assumption 2.1. A **value** is a function $\varphi : G \rightarrow \Re^N$. For each $v \in G$ a value assigns a unique payoff vector in $\Re^N$. For any $v \in G$ let the set $K(v)$ be

$$K(v) = \{S \subseteq N : v(S) \neq \varnothing\}.$$

The set $K(v)$ is the set of **known coalitions** in the game v. Two games $v, w \in G$ will be called **comparable** if $K(v) = K(w)$. Let $K_i(v)$ be the set

$$K_i(v) = \{S \in K(v) : i \in S\}.$$

The set $K_i(v)$ contains those coalitions of which player i is a member and whose worths are known in the game v. Note that Assumption 2.1 ensures that for every $v \in G$, $|K_i(v)| \geq 1$. The following example should help to make these ideas clear.

2.1. An illustrative example

Suppose the set of players is $N = \{1, 2, 3\}$. The worth of the grand coalition is $v(N) = 8$. The worths of the smaller coalitions are

$$v(12) = v(13) = 8, \quad v(1) = 8, \quad v(\varnothing) = 0$$

and

$$v(S) = \varnothing \quad \text{for all other coalitions.}$$

How should the money worth of the grand coalition, 8, be distributed amongst the players? The payoff vector which this work proposes to the distribution problem is $(x_1, x_2, x_3) = (4, 2, 2)$. This distribution seems fair because there are more coalitions which contain player 1 whose worths are known. Also, there are the same number of known coalitions containing players 2 and 3, so it seems reasonable to give players 2 and 3 the same payoff.

2.2. The value and axioms

Here four axioms are defined: efficiency, additivity, nullifying player and known coalitions. The first is efficiency, which requires that the value always distributes all of the worth of the grand coalition amongst the players.

Definition 2.1. A value φ satisfies efficiency if $\sum_{i \in N} \varphi_i(N, v) = v(N)$ for every $v \in G$.

The axiom of additivity links the payoffs which players receive across the addition of different games. Before defining the axiom of additivity, let us define the sum of games on the missing information domain.

Definition 2.2. Suppose $v, w \in G$. If the games v and w are comparable, so $K(v) = K(w)$, the game $u = v + w$ is given by

$$u(S) = \begin{cases} v(S) + w(S) & \text{if } S \in K(v); \\ \varnothing & \text{if } S \notin K(v). \end{cases}$$

Addition of two games is only defined when the games have the same set of known coalitions.

Definition 2.3. A value φ satisfies additivity if, whenever $v, w \in G$ are comparable, $\varphi_i(N, v+w) = \varphi_i(N, v) + \varphi_i(N, w)$ for every $i \in N$.

A player $i \in N$ in game $v \in G$ is called a **nullifying player** if $v(S) = 0$ for every $S \in K_i(v)$. If a player is a nullifying player then all of the known coalitions which contain the player have a worth of zero. This is an intuitive extension of the nullifying player axiom used in [13] to the missing information domain. For any $v \in G$ let $\mathcal{N}(v)$ denote the set of nullifying players in game v.

Definition 2.4. A value φ satisfies nullifying player if, whenever $i \in \mathcal{N}(v)$, then $\varphi_i(N, v) = 0$.

Finally, the axiom of known coalitions requires that the payoffs of players, who are not nullifying players, be related to the number of known coalitions of which they are members.

Definition 2.5. A value φ satisfies known coalitions if, whenever $i, j \in N \setminus \mathcal{N}(v)$, then

$$|K_j(v)|\varphi_i(N, v) = |K_i(v)|\varphi_j(N, v).$$

The intuition behind the known coalitions axiom is that the payoff of a player, who is not nullifying player, should increase when there is positive worth to distribute and, other things equal, the number of known coalitions containing the player increases. In essence, the axiom is a strengthening of the usual axiom of symmetry. Two players $i, j \in N$ are symmetric in game $v \in G$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for every $S \subseteq N \setminus \{i, j\}$. If two non-nullifying players are symmetric in game v then the axiom of known coalitions ensures that they receive the same payoff. However, the axiom also relates the payoffs of players who may not be symmetric in a game. The value which this work suggests is:

$$\varphi_i^*(N, v) = \frac{|K_i(v)|}{\sum_{j \in N} |K_j(v)|} v(N) \quad \text{for every } i \in N. \tag{1}$$

This value is defined for any game in G and distributes the worth of the grand coalition in such a way that each player receives worth in proportion to the number of known coalitions of which they are a member. If the game $v \in G$ is such that $K(v) = 2^N$ then $\varphi_i^*(N, v) = v(N)/n$ for every $i \in N$. Therefore the value coincides with the equal division solution when all the information is contained in the characteristic function. This is why the value proposed here is egalitarian: it takes into consideration the amount of information contained in the characteristic function, but not what the information reveals about players' claims upon the worth of the grand coalition. The next result demonstrates that this value satisfies the four axioms.

Proposition 2.1. *The value φ^* satisfies efficiency, additivity, nullifying player and known information.*

Proof. It is evident from the definition of φ^* that it satisfies efficiency and additivity. To see that φ^* satisfies nullifying player, suppose $i \in \mathcal{N}(v)$. Then $v(N) = 0$ and

$$\varphi_i^*(N, v) = \frac{|K_i(v)| \cdot 0}{\sum_{j \in N} |K_j(v)|} = 0.$$

Finally, to see that φ^* satisfies known information, suppose $i, j \in N \setminus \mathcal{N}(v)$. Then

$$|K_j(v)|\varphi_i^*(N, v) = \frac{|K_j(v)||K_i(v)|v(N)}{\sum_{l \in N} |K_l(v)|} = |K_i(v)|\varphi_j^*(N, v). \quad \blacksquare$$

For any $v \in G$, and $T \in K(v)$ with $T \neq \varnothing$, consider the following special games:

$$w_{T,K(v)}(S) = \begin{cases} 1 & \text{if } S = T, \quad S \in K(v); \\ 0 & \text{if } S \neq T, \quad S \in K(v); \\ \varnothing & \text{if } S \notin K(v). \end{cases}$$

For any $\alpha \in \Re$ the game $\alpha w_{T,K(v)}$ is given by:

$$\alpha w_{T,K(v)}(S) = \begin{cases} \alpha & \text{if } S = T, \quad S \in K(v); \\ 0 & \text{if } S \neq T, \quad S \in K(v); \\ \varnothing & \text{if } S \notin K(v). \end{cases}$$

Thinking of a game v as a vector in $\Re^{K(v)\setminus\{\varnothing\}}$, the next result states that the special games above are a basis of the space $\Re^{K(v)\setminus\{\varnothing\}}$.

Proposition 2.2. *For each $v \in G$ there exist unique real numbers $\{\alpha_T : T \in K(v) \setminus \{\varnothing\}\}$ such that*

$$v(S) = \sum_{T \in K(v)\setminus\{\varnothing\}} \alpha_T w_{T,K(v)}(S) \quad \textit{for every } S \in K(v).$$

Proof. The result follows from noting that the games $w_{T,K(v)}$ are the standard canonical basis on $\Re^{K(v)\setminus\{\varnothing\}}$. $\blacksquare$

Using this result, the final proposition shows that φ^* is the only value to satisfy the four axioms on the space G.

Proposition 2.3. *The unique value satisfying efficiency, additivity, nullifying player and known coalitions on G is φ^*.*

Proof. As φ^* satisfies the four axioms, all that needs to be demonstrated is that the four axioms uniquely determine the payoff vector in any game.

Let φ be a value satisfying the four axioms. Fix a $v \in G$. From Proposition 2.2, and as φ satisfies additivity,

$$\varphi(N, v) = \sum_{T \in K(v)\setminus\{\varnothing\}} \varphi(N, \alpha_T w_{T,K(v)}).$$

If $\alpha_T = 0$ then every player in $\alpha_T w_{T,K(v)}$ is a nullifying player and $\varphi_i(N, \alpha_T w_{T,K(v)}) = 0$ for every $i \in N$.

If $\alpha_T \neq 0$ and $i \in N \setminus T$ then i is a nullifying player in the game $\alpha_T w_{T,K(v)}$. Therefore

$$\varphi_i(N, \alpha_T w_{T,K(v)}) = 0 \quad \text{for every } i \in N \setminus T.$$

The axiom of known coalitions requires, for every $i, j \in T$, that

$$|K_j(\alpha_T w_{T,K(v)})|\varphi_i(N, \alpha_T w_{T,K(v)}) \\ = |K_i(\alpha_T w_{T,K(v)})|\varphi_j(N, \alpha_T w_{T,K(v)}).$$

As $|K_j(\alpha_T w_{T,K(v)})| = |K_j(v)|$ and $|K_i(\alpha_T w_{T,K(v)})| = |K_i(v)|$ we have

$$|K_j(v)|\varphi_i(N, \alpha_T w_{T,K(v)}) = |K_i(v)|\varphi_j(N, \alpha_T w_{T,K(v)}).$$

Fixing i and summing over all $j \in T$ yields

$$\left(\sum_{j \in T} |K_j(v)|\right) \varphi_i(N, \alpha_T w_{T,K(v)}) \\ = |K_i(v)| \left(\sum_{j \in T} \varphi_j(N, \alpha_T w_{T,K(v)})\right).$$

As φ satisfies efficiency, $\sum_{j \in T} \varphi_j(N, \alpha_T w_{T,K(v)}) = \alpha_T w_{T,K(v)}(N)$. Hence

$$\left(\sum_{j \in T} |K_j(v)|\right) \varphi_i(N, \alpha_T w_{T,K(v)}) = |K_i(v)|\alpha_T w_{T,K(v)}(N)$$

and

$$\varphi_i(N, \alpha_T w_{T,K(v)}) = \frac{|K_i(v)|}{\sum_{j \in T} |K_j(v)|} \alpha_T w_{T,K(v)}(N) \\ \text{for every } i \in T.$$

Therefore the four axioms uniquely determine the payoff vector in each game. $\blacksquare$

3. Concluding remarks

This work has suggested a value for a cooperative game with missing information. The value φ^* has the attractive properties that it can be applied to any game, is easy to calculate and has intuitive axiomatic characterization. However, there are two weaknesses which it would be desirable to remedy. First, in a cooperative game with missing information one would like to reward the players for two things: (i) the amount of information there is about the players (the number of coalitions there are containing the players) (ii) the actual worths of coalitions and what they reveal about the marginal contributions of the players. Clearly, φ^* only rewards the players for (i) and does not take (ii) into consideration. When all the information is contained in the characteristic function (i) does not have to taken into consideration and one can reward players solely on the basis of (ii). This is precisely what the Shapley value does. But, it would be desirable to find a value which is versatile enough to reward players on the basis of both (i) and (ii). Such a value would inevitably be more complicated than φ^*, and as a result, it may not be easy to provide an axiomatic characterization. Second, the value φ^* does not satisfy linear covariance of games. This is a general weakness of egalitarian solutions and was noted by [13]. It would be useful to find values for cooperative games with missing information which satisfy linear covariance and can also be characterized axiomatically, such as an analogue of the centre of the imputation set. How this set should be extended to the model of a cooperative game with missing information is an interesting open question.

Acknowledgements

The author would like to thank the attendees of the Liverpool economic research seminar and colleagues at the University of Liverpool, especially Christian Bach, Gary Cook, Dominique Demougin, Yiquan Gu, Sanna Laksá, Eunyoung Moon, Harvey Upton and Yunchou Wu for their encouragement and helpful comments during the writing of this paper.

References

[1] R. Aumann, Values of markets with a continuum of traders, Econometrica 32 (1975) 39–50.
[2] A. Casajus, Monotonic redistribution of performance-based allocations: A case for proportional taxation, Theor. Econ. 10 (2015) 887–892.
[3] D. Granot, Cooperative games in stochastic characteristic function form, Manage. Sci. 27 (1977) 621–630.
[4] H. Habis, P.J.J. Herings, Transferable utility games with uncertainty, J. Econom. Theory 146 (5) (2011) 2126–2139.
[5] S. Laksá, R.R. Routledge, Information and the Value of a Game, University of Liverpool Discussion Paper, 2015.
[6] M. Malawski, Procedural values for cooperative games, Internat. J. Game Theory 42 (2013) 305–324.
[7] M. Maschler, E. Solan, S. Zamir, Game Theory, Cambridge University Press, Cambridge, 2013.
[8] A. Nowak, T. Radzik, A solidarity value for n-person transferable utility games, Internat. J. Game Theory 23 (1994) 43–48.
[9] R.R. Routledge, Deviations, uncertainty and the core, Games Econom. Behav. 88 (2014) 286–297.
[10] D. Schmeidler, The nucleolus of a characteristic function game, SIAM J. Appl. Math. 17 (6) (1969) 1163–1170.
[11] L.S. Shapley, A value for n-person games, in: Alvin E. Roth (Ed.), The Shapley Value Essays in Honor of Lloyd S. Shapley, Cambridge University Press, 1953, (reprint).
[12] J. Suijs, Nucleoli for stochastic cooperative games, in: Cooperative Decision-Making Under Risk, Springer, 2000.
[13] R. van den Brink, Null or nullifying players: The difference between the shapley value and equal division solutions, J. Econom. Theory 136 (1) (2007) 767–775.
[14] H.P. Young, Monotonic solutions of cooperative games, Internat. J. Game Theory 14 (2) (1985) 65–72.
[15] H.P. Young, Individual contribution and just compensation, in: Alvin E. Roth (Ed.), The Shapley Value Essays in Honor of Lloyd S. Shapley, Cambridge University Press, 1988.